The Price to Pay for Forgoing Normalization in Fair Division of Indivisible Goods

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Abstract

We study the complexity of fair division of indivisible goods and consider settings where agents can have nonzero utility for the empty bundle. This is a deviation from a common normalization assumption in the literature, and we show that this inconspicuous change can lead to an increase in complexity: In particular, while an allocation maximizing social welfare by the Nash product is known to be easy to detect in the normalized setting whenever there are as many agents as there are resources, without normalization it can no longer be found in polynomial time, unless $\mathrm{P} = \mathrm{NP}$. The same statement also holds for egalitarian social welfare. Moreover, we show that it is NP-complete to decide whether there is an allocation whose Nash product social welfare is above a certain threshold if the number of resources is a multiple of the number of agents. Finally, we consider elitist social welfare and prove that the increase in expressive power by allowing negative coefficients again yields NP-completeness.

Introduction

We consider problems of social welfare optimization for allocating indivisible resources (or goods or objects or items) and study them in terms of their computational complexity. For an overview of the field, see the survey by Chevaleyre et al. (2006) and the book chapters by Bouveret, Chevaleyre, and Maudet (2016) and by Lang and Rothe (2015).

We restrict our attention to k-additive utilities. A common assumption in the literature is that an agent that receives no resources should have utility zero. We deviate from this normalization assumption. For a fixed number of agents and resources, this allows for more opportunities to increase social welfare because agents can forgo resources. While it may happen that some agent has to receive a certain resource under the normalization assumption in order to guarantee some utility level among all agents, it is now possible that a greater utility level is achievable by assigning no resources to some agents. The excess resources can then be allocated to other agents.

Note that performing a simple "shift" to convert nonnormalized utility functions to normalized utility functions does not capture the allocation model. In order to simulate the fact that no resources can be assigned to an agent while still realizing positive utility, additional resources have to be introduced into the original model. This is problematic for the setting where there are as many agents as resources, which we are going to consider because this is one of the few settings where polynomial-time algorithms do exist.

Our contribution is to show that allowing nonnormalized utility functions comes at a steep cost, namely, that it is unlikely that polynomial-time algorithms for maximizing social welfare exist. This is in contrast to the setting of normalized utility functions, where, under certain restrictions, such algorithms exist.

We also consider elitist social welfare that can be maximized in polynomial time for k-additive utilities whenever all coefficients in the k-additive representation are nonnegative. We show that such an algorithm cannot exist for the same problem under k-additive utility functions for $k \geq 2$ and *arbitrary* coefficients, assuming $\mathrm{P} \neq \mathrm{NP}$. This is based on a reduction that Chevaleyre et al. (2008) designed for utilitarian social welfare.

Preliminaries

Let A denote a set of n *agents* and R a set of m indivisible and nonshareable *resources*. Each agent $a_i \in A$ is equipped with a utility function $u_i : 2^R \to \mathbb{Q}$ and $U = (u_1, \ldots, u_n)$. Then (A, R, U) is an *allocation setting*. A utility function u over resources R is *k-additive* if for every $X \subseteq R$ there is a (unique) coefficient $\alpha^X \in \mathbb{Q}$, which vanishes if $\|X\| > k$, such that for every $Y \subseteq R$,

$$u(Y) = \sum_{X \subseteq Y} \alpha^X.$$

In an allocation setting (A, R, U), an *allocation* π is a partition of R into $n = \|A\|$ (possibly empty) subsets. Then $\pi(a_i)$ denotes the bundle that agent a_i receives. Denote by $\Pi_{A,R}$ the set of all allocations for agents A and resources R.

We measure the social welfare of an allocation π using

- *utilitarian social welfare*:
$$sw_u(\pi) = \sum_{a_i \in A} u_i(\pi(a_i)),$$
- *egalitarian social welfare*:
$$sw_e(\pi) = \min_{a_i \in A} u_i(\pi(a_i)),$$

- *Nash product social welfare*:

$$sw_N(\pi) = \prod_{a_i \in A} u_i(\pi(a_i)), \text{ and}$$

- *elitist social welfare*:

$$sw_E(\pi) = \max_{a_i \in A} u_i(\pi(a_i)).$$

Utilitarian social welfare, sw_u, captures the average utility that agents receive in an allocation setting. Clearly, lopsided allocations are possible when a single agent receives all the goods. This is put to an extreme under elitist social welfare, sw_E, whose usage can be justified, e.g., in settings where the center controls all agents. At the other side of the spectrum is egalitarian social welfare, sw_e. Maximizing egalitarian social welfare corresponds to paying attention to the worst-off agent only, neglecting concerns of efficiency. Nash product social welfare, sw_N, strikes a balance between sw_u and sw_e in the sense that balanced utility values maximize sw_N and its outcomes are Pareto-efficient (see also the paper by Caragiannis et al. (2016)).

Let us now define our optimization problems and their associated decision problems, starting with the most prominent one: the problem of maximizing utilitarian social welfare.

$\mathbb{Q}$-MAXIMUM-UTILITARIAN-SOCIAL-WELFARE$_{k\text{-ADD}}$	
Input:	An allocation setting (A, R, U), where each utility function $u_i : 2^R \rightarrow \mathbb{Q}$ is represented in k-additive form.
Output:	$\max\{sw_u(\pi) \mid \pi \in \Pi_{A,R}\}$

We will also use the shorthand $\mathbb{Q}$-MAX-USW$_{k\text{-ADD}}$ for this problem. If we require in addition that the number of agents be equal to the number of resources, the resulting problem is denoted by $\mathbb{Q}$-MAX-USW$^{n=m}_{k\text{-ADD}}$; analogously, this superscript "$n = m$" indicates the same restriction for the problems defined below.

The decision problem associated with the optimization problem $\mathbb{Q}$-MAX-USW$_{k\text{-ADD}}$ is defined as follows.

$\mathbb{Q}$-UTILITARIAN-SOCIAL-WELFARE-OPTIMIZATION$_{k\text{-ADD}}$	
Given:	An allocation setting (A, R, U), where each utility function $u_i : 2^R \rightarrow \mathbb{Q}$ is represented in k-additive form, and a number $K \in \mathbb{N}$.
Question:	Does there exist an allocation $\pi \in \Pi_{A,R}$ such that $sw_u(\pi) \geq K$?

Again, we will also use the shorthand $\mathbb{Q}$-USWO$_{k\text{-ADD}}$ for this problem. Furthermore, by replacing utilitarian social welfare by other types of social welfare, we can define the following decision and optimization problems. Here, the symbol $\mathbb{Q}^+$ denotes the set of nonnegative rational numbers.

- $\mathbb{Q}$-EGALITARIAN-SOCIAL-WELFARE-OPTIMIZATION$_{k\text{-ADD}}$ (for short, $\mathbb{Q}$-ESWO$_{k\text{-ADD}}$) and $\mathbb{Q}$-MAX-EGALITARIAN-SOCIAL-WELFARE$_{k\text{-ADD}}$ (for short, $\mathbb{Q}$-MAX-ESW$_{k\text{-ADD}}$),
- $\mathbb{Q}^+$-NASH-PRODUCT-SOCIAL-WELFARE-OPTIMIZATION$_{k\text{-ADD}}$ (for short, $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$) and $\mathbb{Q}^+$-MAX-NASH-PRODUCT-SOCIAL-WELFARE$_{k\text{-ADD}}$ (for short, $\mathbb{Q}^+$-MAX-NPSW$_{k\text{-ADD}}$), and
- $\mathbb{Q}$-ELITIST-SOCIAL-WELFARE-OPTIMIZATION$_{k\text{-ADD}}$ (for short, $\mathbb{Q}$-ELSWO$_{k\text{-ADD}}$) and $\mathbb{Q}$-MAX-ELITIST-SOCIAL-WELFARE$_{k\text{-ADD}}$ (for short, $\mathbb{Q}$-MAX-ELSW$_{k\text{-ADD}}$).

We assume the reader to be familiar with the basic notions of complexity theory, such as the complexity classes P (deterministic polynomial time) and NP (nondeterministic polynomial time), polynomial-time many-one reducibility, and the notions of NP-hardness and -completeness based on this reducibility.

Nash Product Social Welfare

In this section, we study the complexity of social welfare optimization by the Nash product, assuming k-additive utility functions for $k \geq 1$.

Known Results

Roos and Rothe (2010) showed that the general problem $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ is NP-complete. NP-completeness still holds when the given allocation setting has only two agents and normalized utility functions. In addition, they and, independently, Ramezani and Endriss (2010) showed that this problem is NP-complete also when utilities are given in the bundle form (Roos and Rothe 2010; Nguyen et al. 2014). Also for other representation forms that we do not consider here, analogous results have been obtained (Ramezani and Endriss 2010) (see also, e.g., (Cole et al. 2017) for the approximability of Nash product social welfare).

NP-hardness of $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ rests on a reduction from the problem PARTITION that is well known to be NP-complete (Karp 1972).

PARTITION	
Given:	A sequence $(c_1, \ldots, c_s)$ of nonnegative integers such that $C = \sum_{i=1}^{s} c_i$ for an even number $C \in \mathbb{N}$.
Question:	Does there exist a subset $J \subseteq S = \{1, \ldots, s\}$ such that $\sum_{i \in J} c_i = \sum_{i \in S \setminus J} c_i$?

Regarding the optimization problem, Nguyen, Roos, and Rothe (2013) proposed a polynomial-time algorithm that provides an allocation with maximal Nash product if both the number of agents equals the number of resources to distribute and the utility functions are normalized.

The Price to Pay for Forgoing Normalization

We show that, assuming P $\neq$ NP, $\mathbb{Q}^+$-NPSWO$^{n=m}_{k\text{-ADD}}$ is no longer solvable in polynomial time if utility functions are not required to be normalized, i.e., if $u_i(\emptyset) = \lambda_i$ with $\lambda_i \in \mathbb{Q}^+ \setminus \{0\}$ for at least one agent a_i. Concretely, we show NP-completeness of $\mathbb{Q}^+$-NPSWO$^{n=m}_{k\text{-ADD}}$. Our proof is based on the observation that in the construction of Roos and Rothe (2010) arbitrarily many agents may be added to the given allocation setting without changing its Nash product. In this sense, it can be seen as an extension of their proof of NP-completeness of $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$.

Theorem 1. *For each* $k \geq 1$, $\mathbb{Q}^+$-NPSWO$^{n=m}_{k\text{-ADD}}$ *is* NP-*complete.*

Proof. Membership of $\mathbb{Q}^+$-NPSWO$^{n=m}_{k\text{-ADD}}$ in NP is easy to observe. To prove its NP-hardness, we focus on the case $k = 1$ (the cases $k > 1$ then follow immediately) and we give a reduction from the problem PARTITION. Let $c = (c_1, \ldots, c_s)$ be a sequence of nonnegative integers such that $C = \sum_{i=1}^{s} c_i$ is even. Let $S = \{1, \ldots, s\}$. Construct a $\mathbb{Q}^+$-NPSWO$^{n=m}_{1\text{-ADD}}$ instance $((A, R, U), K)$ from (c, C) with s agents, s resources, and s utility functions having only nonnegative coefficients α_i^T for any bundle $T \subseteq R$. Specifically, let $A = \{a_1, \ldots, a_s\}$ and $R = \{r_1, \ldots, r_s\}$, and define the utility functions u_i by the following coefficients:

$$\alpha_i^{\{r_j\}} = c_j \qquad (i \in \{1, 2\})$$
$$\alpha_i^{\emptyset} = 1 \qquad (i \in \{3, \ldots, s\})$$

for $1 \leq j \leq s$. Since we assume the 1-additive case, all other coefficients are zero. For the lower bound, choose $K = ({}^{C}\!/_{2})^2$.

It remains to show that (c, C) is a yes-instance of PARTITION if and only if $((A, R, U), K)$ is a yes-instance of $\mathbb{Q}^+$-NPSWO$^{n=m}_{1\text{-ADD}}$.

From left to right, suppose that (c, C) is a yes-instance of PARTITION. Thus there is a subset $J \subseteq S$ such that $\sum_{i \in J} c_i = \sum_{i \in S \setminus J} c_i = {}^{C}\!/_{2}$. Define an allocation π as follows:

$$\pi(a) = \begin{cases} \{r_j \mid j \in J\} & \text{if } a = a_1, \\ \{r_j \mid j \in S \setminus J\} & \text{if } a = a_2, \\ \emptyset & \text{if } a \in \{a_3, \ldots, a_s\}. \end{cases}$$

It follows that

$$\begin{aligned} sw_N(\pi) &= u_1(\pi(a_1))u_2(\pi(a_2)) \cdots u_s(\pi(a_s)) \\ &= \left(\sum_{i \in J} u_1(\{r_i\})\right) \left(\sum_{j \in S \setminus J} u_2(\{r_j\})\right) 1^{(s-2)} \\ &= \left(\sum_{i \in J} c_i\right) \left(\sum_{j \in S \setminus J} c_j\right) \\ &= \left(\frac{C}{2}\right)^2 = K. \end{aligned}$$

From right to left, suppose that (c, C) is a no-instance of PARTITION. By definition of the coefficients α_i^T we have $u_i(B) = 1$ for $3 \leq i \leq s$ and all bundles $B \subseteq R$. Suppose there were an allocation π satisfying $sw_N(\pi) \geq K$. For convenience, view $sw_N(\pi)$ as a function

$$g : [0, C] \times [0, C] \to \mathbb{Q}, \qquad (x, y) \mapsto x \cdot y \cdot 1^{s-2},$$

where $x + y = \xi$ for some $\xi \in [0, C] \cap \mathbb{N}$.

Substituting the constraints simplifies this function to:

$$h_\xi : [0, C] \to \mathbb{Q}, \qquad x \mapsto x \cdot (\xi - x).$$

The first two derivatives of h_ξ are given by

$$\frac{\partial h_\xi}{\partial x}(x) = -2x + \xi \qquad \text{and} \qquad \frac{\partial^2 h_\xi}{\partial^2 x}(x) = -2 < 0.$$

For $0 \leq \xi \leq C$, we have $h_\xi(0) = 0$ and

$$h_\xi(C) = C \cdot (\xi - C) = \xi C - C^2 \leq 0.$$

It follows that the social welfare is maximal for $x = {}^{\xi}\!/_{2}$. Since the value of h_ξ at this point is $h_\xi({}^{\xi}\!/_{2}) = ({}^{\xi}\!/_{2})^2$, $({}^{\xi}\!/_{2})^2$ is monotonically increasing on $\mathbb{Q}^+$, and ξ is bounded above by C, the partition must satisfy $u_1(\pi(a_1)) = u_2(\pi(a_2)) = {}^{C}\!/_{2}$ for reaching the bound K. Since all c_i are nonnegative, this means that we must have started from a yes-instance of PARTITION, a contradiction. It follows that no allocation π can satisfy $sw_N(\pi) \geq K$.

Since the transformation can be computed in polynomial time, NP-completeness of $\mathbb{Q}^+$-NPSWO$^{n=m}_{k\text{-ADD}}$ follows. ❑

Example 1. *Let* $c = (c_1, \ldots, c_5) = (1, 2, 3, 5, 9)$ *and* $C = \sum_{i=1}^{5} c_i = 20$ *be given. Construct from* (c, C) *the* $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ *instance* $((A, R, U), K)$ *according to the reduction given in the proof of Theorem 1, with lower bound* $K = ({}^{20}\!/_{2})^2 = 100$*, agents* $A = \{a_1, a_2, \ldots, a_5\}$*, resources* $R = \{r_1, \ldots, r_5\}$*, and utility functions* $u_i(B) = 1[r_1] + 2[r_2] + 3[r_3] + 5[r_4] + 9[r_5]$ *for* $i \in \{1, 2\}$*, and* $u_j(B) = 1[\emptyset]$ *for* $j \in \{3, 4, 5\}$*, where* $[r_i]$ *is 1 if* $\{r_i\} \subseteq B$ *and is* 0 *otherwise.*

For $J = \{1, 5\}$*, we have* $\sum_{i \in J} c_i = \sum_{j \in S \setminus J} c_j$*, so this is a yes-instance of* PARTITION.

The allocation from the proof of Theorem 1 is $\pi = (\{r_1, r_5\}, \{r_2, r_3, r_4\}, \emptyset, \emptyset, \emptyset)$ *with* $sw_N(\pi) = 100 \geq K$*, so we also have a yes-instance of* $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$.

The hardness result of Theorem 1 depends on the fact that the number of agents with utility functions that are not normalized is a function of n. Suppose a constant c denotes the number of agents whose utility functions are not normalized. Then, for $n = m$, by exhaustive search we can find an allocation of maximum Nash product social welfare in time $\mathcal{O}(m^{3c}t)$, where t is the polynomial running time of the algorithm by Nguyen, Roos, and Rothe (2013) for maximizing the Nash product under normalized utility functions.

In more detail, the algorithm considers all 2^c possibilities which of the c agents are going to be ignored and therefore receive no goods. Denote by d the number of agents that are not ignored. If an agent i is not ignored, we set $u_i(\emptyset) = 0$. Since $c - d$ agents are ignored, $c - d$ goods have to be merged to supergoods. To achieve this, there are $\binom{m}{c-d}$ ways to choose the $c - d$ goods, and for each such choice there are $(m - (c - d))^{c-d}$ ways to merge the chosen goods to supergoods. Then an agent's utility for a supergood is the utility for the empty bundle plus the utility of the bundle that contains all goods that make up that supergood. In this reduced instance, we have an equal number of agents and goods. Therefore, we can apply the algorithm by Nguyen, Roos, and Rothe (2013) for normalized utility functions. We adjust the social welfare of the output allocation by multiplying by the utility values of the ignored agents for the empty bundle and collect the resulting allocation in a list. After we have considered all 2^c possibilties, we pick an allocation of maximum welfare.

Now, let π^* be an allocation of maximum welfare according to the above procedure. Let $\widehat{\pi}$ be some allocation. Then the Nash product of π^* is no less than the Nash product of $\widehat{\pi}$. If $\widehat{\pi}$ was in the list of the above procedure, the claim holds. If $\widehat{\pi}$ was not in the list, then denote by I the set of ignored agents in $\widehat{\pi}$ (i.e., agents that receive no goods). Consider the suballocation $\widehat{\pi}_I$ that results from $\widehat{\pi}$ by removing the entries of the ignored agents in I. Some agents receive only one good, whereas other agents receive multiple goods. There is an iteration in the above algorithm where agents in I are ignored and supergoods are formed according to the bundles in $\widehat{\pi}_I$ that contain multiple goods. Since $\widehat{\pi}$ is not in the list, the matching-based algorithm by Nguyen, Roos, and Rothe (2013) finds an allocation π'_I with greater or equal social welfare, i.e., $sw_N(\pi'_I) \geq sw_N(\widehat{\pi}_I)$. After adjusting for the ignored agents, the resulting allocation π' is added to the list. Because the set of ignored agents is the same for π'_I and $\widehat{\pi}_I$, the social welfare of $\widehat{\pi}_I$ cannot become greater than the social welfare of π'_I after the adjustment. Overall, we have $sw_N(\pi^*) \geq sw_N(\pi') \geq sw_N(\widehat{\pi})$. Hence, we have shown the following result:

Theorem 2. *Let c be the number of agents with nonnormalized utility functions. For $m \geq 2$, $\mathbb{Q}^+$-MAX-NPSW$^{n=m}_{\text{1-ADD}}$ can be solved in time $\mathcal{O}(m^{3c}p)$, where p is a polynomial function.*

Further Restrictions

The problem $\mathbb{Q}^+$-NPSWO$^{n=m}_{k\text{-ADD}}$ is a special case of $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$, so NP-hardness of the former is immediately inherited by the latter, and this also holds true for only two agents (Roos and Rothe 2010, Theorem 5.1).

We now consider the case where the number of resources to distribute is a multiple of the number of agents.

Theorem 3. *Fix an integer $p \geq 2$. For each $k \geq 1$, the problem $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ restricted to instances with $\|R\| = p \cdot \|A\|$ and normalized utility functions is* NP-*complete.*

Proof. Membership of the problem in NP is again easy to observe, just as in the proof of Theorem 1. To prove its NP-hardness, it is sufficient to consider the case with $k = 1$ and $p = 2$.

Again, we give a reduction from PARTITION. Let $(c_1, \ldots, c_s)$ be a sequence of nonnegative integers such that $C = \sum_{i=1}^{s} c_i$ is even. Let $S = \{1, \ldots, s\}$. Construct a $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ instance $I = ((A, R, U), K))$ with s agents, $2s$ resources, and s utility functions having only nonnegative coefficients α_i^T for any bundle $T \subseteq R$:

$$\begin{aligned} A &= \{a_1, a_2, a_3, \ldots, a_s\}, \\ R &= \{r_1^1, \ldots, r_s^1, r_1^2, \ldots, r_s^2\}, \text{ and} \\ K &= \left(\frac{C}{2}\right)^2. \end{aligned}$$

Define the utility functions u_i by the following coefficients (and set all other coefficients to zero):

$$\begin{aligned} &\alpha_i^{\{r_j^1\}} = c_j \qquad \alpha_i^{\{r_j^2\}} = 0 \qquad (1 \leq i \leq 2, \quad 1 \leq j \leq s) \\ &\alpha_i^{\{r_\ell^1\}} = 0 \qquad \alpha_i^{\{r_\ell^2\}} = 1 \qquad (3 \leq i \leq s, \quad 3 \leq \ell \leq s) \end{aligned}$$

It follows that $\alpha_q^{\{r_1^2\}} = \alpha_q^{\{r_2^2\}} = 0$ for all q, $1 \leq q \leq s$.

We claim that (c, C) is a yes-instance of PARTITION if and only if $((A, R, U), K)$ is a yes-instance of $\mathbb{Q}^+$-NPSWO$^{n=m}_{\text{1-ADD}}$.

From left to right, suppose that I is a yes-instance of PARTITION. Then there is a subset $J \subseteq S$ such that $\sum_{i \in J} c_i = \sum_{i \in S \setminus J} c_i = C/2$. Define an allocation π as follows:

$$\pi(a) = \begin{cases} \{r_j^1 \mid j \in J\} \cup \{r_1^2, r_2^2\} & \text{if } a = a_1, \\ \{r_j^1 \mid j \in S \setminus J\} & \text{if } a = a_2, \\ \{r_t^2\} & \text{if } a = a_t \ (3 \leq t \leq s). \end{cases}$$

It follows that

$$\begin{aligned} sw_N(\pi) &= u_1(\pi(a_1))\, u_2(\pi(a_2)) \cdots u_s(\pi(a_s)) \\ &= \left(u_1(\{r_1^2\}) + u_1(\{r_2^2\}) + \sum_{i \in J} u_1(\{r_i^1\})\right) \cdot \\ &\quad \left(\sum_{j \in S \setminus J} u_2(\{r_j^1\})\right) \cdot 1^{(s-2)} \\ &= \left(\sum_{i \in J} c_i\right) \cdot \left(\sum_{j \in S \setminus J} c_j\right) \\ &= \left(\frac{C}{2}\right)^2 = K. \end{aligned}$$

From right to left, suppose that I is a no-instance of PARTITION. Assume there were an allocation π satisfying $sw_N(\pi) \geq K$. Then we have

$$sw_N(\pi) = x \cdot y \cdot \left(\prod_{j=3}^{s} (z_j \cdot 1)\right) \geq K,$$

where z_j indicates for each agent a_j how many resources from the set $R = \{r_3^2, \ldots, r_s^2\}$ are assigned to her. Since the utility functions are normalized, every agent a_q with $3 \leq q \leq s$ must be assigned exactly one resource from R for the Nash product to be distinct from zero.

Because of $u_q(\{r_1^2\}) = u_q(\{r_2^2\}) = 0$ for $1 \leq q \leq s$ and $u_p(\{r_\ell^1\}) = 0$ for $3 \leq p \leq s$ and $1 \leq \ell \leq s$, it suffices to show

$$sw_N(\pi) = x \cdot y$$

for $x + y = \xi \in [0, C] \cap \mathbb{N}$. The same argument as in the proof of Theorem 1 shows that we would have started from a yes-instance of PARTITION then, a contradiction. It follows that no allocation π can satisfy $sw_N(\pi) \geq K$.

Again, the transformation can be computed in polynomial time, which completes the proof of NP-completeness. ❑

Example 2. *Let $c = (c_1, \ldots, c_4) = (1, 3, 5, 7)$ and $C = 16$ be given. Construct from (c, C) the $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$ instance $((A, R, U), K)$ according to the reduction given in the proof of Theorem 3, with lower bound $K = (16/2)^2 = 64$, agents $A = \{a_1, a_2, \ldots, a_4\}$, resources $R = \{r_1^1, \ldots, r_4^1, r_1^2, \ldots, r_4^2\}$, and utility functions $u_i(B) = 1[r_1^1] + 3[r_2^1] + 5[r_3^1] + 7[r_4^1]$ for $i \in \{1, 2\}$, and $u_j(B) = 1[r_3^2] + 1[r_4^2]$ for $j \in \{3, 4\}$. This is a yes-instance of PARTITION, as with $J = \{1, 4\}$ and $S \setminus J = \{2, 3\}$ we have the equality $\sum_{i \in J} c_i = \sum_{j \in S \setminus J} c_j$. The allocation from the proof of Theorem 3 then is $\pi = (\{r_1^1, r_4^1, r_1^2, r_2^2\}, \{r_2^1, r_3^1\}, \{r_3^2\}, \{r_4^2\})$, which satisfies $sw_N(\pi) = 64 \geq K$, so we also have a yes-instance of $\mathbb{Q}^+$-NPSWO$_{k\text{-ADD}}$.*

Egalitarian Social Welfare

In this section, we study the complexity of egalitarian social welfare optimization, again assuming k-additive utility functions for $k \geq 1$.

Known Results

Based on the work of Irving, Leather, and Gusfield (1987), Golovin (2005) provided an algorithm solving the problem $\mathbb{Q}$-MAX-ESW$^{n=m}_{1\text{-ADD}}$ with normalized utility functions in polynomial time. The paper by Bansal and Sviridenko (2006) provides one of the many approximability results on maximizing egalitarian social welfare, see the survey by Nguyen, Roos, and Rothe (2013) for an overview.

The Price to Pay for Forgoing Normalization

As we did in the previous section for the Nash product, we now investigate whether the normalization requirements for the algorithm mentioned above are necessary. We will show that without this normalization, the corresponding decision problem is NP-complete. Making use of a reduction due to Lipton et al. (2004) for two agents and m resources, we provide a reduction from PARTITION to $\mathbb{Q}$-ESWO$_{k\text{-ADD}}$, again by extending the original allocation setting by a suitable number of dummy agents without changing its egalitarian social welfare.

Theorem 4. *For each $k \geq 1$, $\mathbb{Q}$-ESWO$^{n=m}_{k\text{-ADD}}$ is NP-complete.*

Proof. Membership of $\mathbb{Q}$-ESWO$^{n=m}_{k\text{-ADD}}$ in NP again is obvious for each $k \geq 1$. To prove NP-hardness, we only consider the kase $k = 1$ and reduce PARTITION to $\mathbb{Q}$-ESWO$^{n=m}_{1\text{-ADD}}$. Let $c = (c_1, \ldots, c_s)$ be a sequence of nonnegative integers such that $C = \sum_{i=1}^{s} c_i$ is even. Let $S = \{1, \ldots, s\}$. Construct a $\mathbb{Q}$-ESWO$^{n=m}_{1\text{-ADD}}$ instance $I = ((A, R, U), K)$ from (c, C) with lower bound $K = C/2$, s agents, s resources, and s utility functions having only nonnegative coefficients α_i^T for any bundle $T \subseteq R$, i.e., $A = \{a_1, \ldots, a_s\}$ and $R = \{r_1, \ldots, r_s\}$, and define the utility functions u_i by the following coefficients:

$$\alpha_i^{\{r_j\}} = c_j \qquad (i \in \{1, 2\})$$
$$\alpha_i^{\emptyset} = K \qquad (i \in \{3, \ldots, s\})$$

for $1 \leq j \leq s$. By construction all other coefficients are zero.

This transformation obviously can be done in polynomial time. It remains to show that (c, C) is a yes-instance of PARTITION if and only if $((A, R, U), K)$ is a yes-instance of $\mathbb{Q}$-ESWO$^{n=m}_{1\text{-ADD}}$.

From left to right, suppose that (c, C) is a yes-instance of PARTITION. Thus there is a subset $J \subseteq S$ such that $\sum_{i \in J} c_i = \sum_{i \in S \setminus J} c_i = C/2$. Define an allocation π as follows:

$$\pi(a) = \begin{cases} \{r_j \mid j \in J\} & \text{if } a = a_1, \\ \{r_j \mid j \in S \setminus J\} & \text{if } a = a_2, \\ \emptyset & \text{if } a \in \{a_3, \ldots, a_s\}. \end{cases}$$

It follows that

$$\begin{aligned} sw_e(\pi) &= \min\{u_1(\pi(a_1)), u_2(\pi(a_2)), \ldots, u_s(\pi(a_s))\} \\ &= \min\left\{\sum_{i \in J} u_1(\{r_i\}), \sum_{j \in S \setminus J} u_2(\{r_j\}), K, \ldots, K\right\} \\ &= K. \end{aligned}$$

From right to left, suppose that (c, C) is a no-instance of PARTITION. Then $\sum_{i \in J} c_i \neq \sum_{i \in S \setminus J} c_i$ for every subset $J \subseteq S = \{1, \ldots, s\}$.

Suppose there were an allocation π satisfying $sw_e(\pi) \geq K$. Since $u_i(B) = K$ $(3 \leq i \leq s)$ for all bundles $B \subseteq R$, we have

$$\min\{u_1(\pi(a_1)), u_2(\pi(a_2))\} \geq K.$$

This is possible only in one of the following four cases:

1. $u_1(\pi(a_1)) > K = \frac{C}{2}$ and $u_2(\pi(a_2)) > K = \frac{C}{2}$,
2. $u_1(\pi(a_1)) = K = \frac{C}{2}$ and $u_2(\pi(a_2)) > K = \frac{C}{2}$,
3. $u_1(\pi(a_1)) > K = \frac{C}{2}$ and $u_2(\pi(a_2)) = K = \frac{C}{2}$,
4. $u_1(\pi(a_1)) = K = \frac{C}{2} = u_2(\pi(a_2))$.

If one of the first three cases were to occur, it would follow that

$$\left(\frac{C}{2} + \varepsilon_1\right) + \left(\frac{C}{2} + \varepsilon_2\right) = C + (\varepsilon_1 + \varepsilon_2) > C$$

with positive $\varepsilon_1, \varepsilon_2 \in \mathbb{N}$ for the first case, and $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$ for the second and third case; this is possible only if one resource is assigned more than once. In the fourth case, finally, it would follow that we must have started from a yes-instance of PARTITION, again a contradiction. This completes the proof. ❑

Elitist Social Welfare

Finally, we make a small observation regarding elitist social welfare. Heinen, Nguyen, and Rothe (2015) observed that the problem $\mathbb{Q}$-ELSWO$_{1\text{-ADD}}$ (which is called n-RANK DICTATOR in their paper) can be solved in polynomial time. It is not hard to see that essentially the same argument gives the same result for $\mathbb{Q}$-ELSWO$_{k\text{-ADD}}$ for each $k \geq 2$, provided that all coefficients in the k-additive representation are nonnegative.[1] However, if negative coefficients are allowed, this decision problem turns NP-complete, which follows immediately from a known reduction due to Chevaleyre et al. (2004).

Theorem 5. *For each $k \geq 2$, $\mathbb{Q}$-ELSWO$_{k\text{-ADD}}$ with arbitrary coefficients is* NP*-complete.*

Proof. It is obvious that $\mathbb{Q}$-ELSWO$_{k\text{-ADD}}$ is in NP for each $k \geq 2$: Nondeterministically, choose an allocation π and verify whether $\max\{u_i(\pi(a_i)) \mid a_i \in A\} \geq K$.

To prove NP-hardness of $\mathbb{Q}$-ELSWO$_{2\text{-ADD}}$, we make use of a reduction due to Chevaleyre et al. (2004) who showed NP-hardness of $\mathbb{Q}$-USWO$_{2\text{-ADD}}$ by a reduction from the well-known NP-complete problem MAXIMUM-2-SATISFIABILITY (for short, MAX-2-SAT), which is defined as follows:

MAXIMUM-2-SATISFIABILITY	
Given:	A boolean formula φ in conjunctive normal form, where each clause has exactly two literals, and a nonnegative integer K.
Question:	Does there exist a truth assignment simultaneously satisfying at least K clauses of φ?

Consider the reduction from the proof of (Chevaleyre et al. 2008, Proposition 8), which reduces MAX-2-SAT to $\mathbb{Q}$-USWO$_{2\text{-ADD}}$. That means that a MAX-2-SAT instance (φ, K) is mapped to a $\mathbb{Q}$-USWO$_{2\text{-ADD}}$ instance $((A, R, U), K)$ consisting of one resource for each variable of φ, two agents, a_1 and a_2, with utilities $u_2 \equiv 0$ and u_1 as shown in Table 1 such that when there are T satisfied clauses, there are exactly T additive terms equal to 1.

Clause	2-additive term
$(x_i \vee x_i)$	$1[x_i]$
$(\neg x_i \vee \neg x_i)$	$1 - [x_i]$
$(x_i \vee x_j)$	$[x_i] + [x_j] - [x_i][x_j]$
$(x_i \vee \neg x_j)$	$[x_i] + (1 - [x_j]) - [x_i] \cdot (1 - [x_j])$
$(\neg x_i \vee \neg x_j)$	$(1 - [x_i]) + (1 - [x_j]) - (1 - [x_i]) \cdot (1 - [x_j])$

Table 1: 2-additive terms for u_1 assuming $i \neq j$

Note that utility function u_1 can also have negative coefficients. It holds that $((A, R, U), K)$ is a yes-instance of $\mathbb{Q}$-USWO$_{2\text{-ADD}}$ exactly if there exists an allocation π with $sw_u(\pi) = u_1(\pi(a_1)) + u_2(\pi(a_2)) = u_1(\pi(a_1)) \geq K$, if and only if there exists an allocation π with $sw_E(\pi) = \max\{u_1(\pi(a_1)), u_2(\pi(a_2))\} = \max\{T, 0\} \geq K$, which in turn is equivalent to $((A, R, U), K)$ being a yes-instance of $\mathbb{Q}$-ELSWO$_{2\text{-ADD}}$. Hence MAX-2-SAT reduces to $\mathbb{Q}$-ELSWO$_{2\text{-ADD}}$ in polynomial time. The NP-hardness claim for $\mathbb{Q}$-ELSWO$_{k\text{-ADD}}$, $k > 2$, follows immediately. ❑

[1] In particular, this assumption ensures that every agent realizes the highest utility by receiving all resources.

Conclusions

We have studied the implications of the normalization assumption in fair division of indivisible goods. For the common notions of egalitarian and Nash product social welfare, we have shown that this assumption is crucial to have polynomial-time algorithms in certain settings. The key idea of the NP-hardness proofs for nonnormalized utility functions is that dummy agents can be inserted easily to ensure the cardinality constraint. For $n = m$, the results also suggest that there is no general (for a superconstant number of agents with nonnormalized utility functions) and efficient transformation to simulate allocation settings with nonnormalized utility functions using normalized utility functions only, as this would imply P = NP. This is interesting because assigning nonzero utility to the empty bundle corresponds to merely having a positive base level of happiness. Regarding elitist social welfare, note that the reduction in Theorem 5 can also produce nonnormalized utility functions.

In the future, it might be worthwhile to study the effect of the normalization assumption in settings apart from fair division such as (cooperative) game theory.

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