Sub-committee Approval Voting and Generalized Justified Representation Axioms

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Abstract

Social choice is replete with various settings including single-winner voting, multi-winner voting, probabilistic voting, multiple referenda, and public decision making. We study a general model of social choice called sub-committee voting (SCV) that simultaneously generalizes these settings. We then focus on sub-committee voting with approvals and propose extensions of the justified representation axioms that have been considered for proportional representation in approval-based committee voting. We study the properties and relations of these axioms. For each of the axioms, we analyse whether a representative committee exists and also examine the complexity of computing and verifying such a committee.

Introduction

Social choice is a general framework of preference aggregation in which voters express preferences over outcomes and a desirable outcome is selected based on the preferences of the voters (Aziz et al. 2017a; Conitzer 2010). It has also been motivated as being especially relevant to ethical and principled decision making in multi-agent settings (Conitzer et al. 2017). The most classic model of social choice is *(single winner) voting* in which voters express preferences over a set of alternatives and exactly one alternative is selected (Brams and Fishburn 2002). A natural generalization of the model is *muti-winner voting* or *committee voting* in which a set of alternatives is selected (Faliszewski et al. 2017). Another model is *multiple referenda* in which voters vote over multiple but independent binary decisions (Brams, Kilgour, and Zwicker 1997). Probabilistic versions of single-winner voting have also been examined (Gibbard 1977).

In this paper, we study a natural model of social choice that simultaneously generalizes all the social choice settings mentioned above. The advantage of considering a more general combinatorial model (Lang and Xia 2016) is that instead of coming up with desirable axioms, rules, and algorithms in a piecemeal manner for different settings, one can design or apply general principles and approaches that may be compelling for a wide range of settings. Of course certain axioms may only be meaningful for a certain subsetting but as we show in this paper, a positive algorithmic or axiomatic

result for well-justified axioms can be viewed favourably for all relevant sub-settings as well. Another advantage of formalising a general model is that it provides an opportunity to unify different strands of work in social choice. Our model also helps approach the committee voting problem in which there are additional diversity constraints possibly relating to gender, race or skill. Finally, our model applies to general participatory budgeting scenarios (Cabannes 2004) where multiple decisions needs to be made and the minority representation needs to be protected.

After formalizing the SCV (sub-committee voting) setting, we focus on a particular restriction of SCV in which agents or voters only express approvals over some of the alternatives or candidates. The restriction to approvals is desirable because approvals capture dichotomous/binary preferences that are prevalent in many natural settings. Secondly, ordinal and cardinal preferences coincide when preferences are dichotomous. This is desirable since elicitation of cardinal utilities has been considered controversial in decisions concerning public goods.

SCV with approvals can be viewed as a multidimensional generalization of approval-based committee voting (Kilgour 2010). For approval-based committee voting, a particularly appealing axiom that captures representation is justified representation (JR) that requires that a set of voters that is large enough and cohesive enough in their preferences should get at least one approved candidate in the selected committee. The axiom has received considerable attention (Brill et al. 2017; Sánchez-Fernández, Fernández, and Fisteus 2016; Sánchez-Fernández et al. 2017). For SCV with approvals, we extend the justified representation axiom (Aziz et al. 2017b) that has only been studied in the context of committee voting.

One interesting application captured by this SCV framework, which is not possible under standard models, is committee voting in the presence of diversity constraints or quotas. Considering this application highlights the conflict between diversity constraints and the original axioms of fair, or justified, representation. As will be shown this conflict leads to conceptual issues of what is the 'appropriate' generalisation of the JR axiom for SCV instances and also technical issues such as existence and computational intractability of achieving certain axioms whilst diversity constraints are enforced.

Contributions Our contributions are threefold with the first two being conceptual contributions.

Firstly, we study a natural model of social choice called SCV (sub-committee voting) that simultaneously generalizes several previously studied settings.

Secondly, we focus on approval-based SCV and present new notions of justified representation (JR) concepts including Intra-wise JR (IW-JR) and Span-wise JR (SW-JR). These distinct notions lead to 'local' and 'global' approaches to representation, respectively. These notions address the push in AI to formalize fairness in decision-making scenarios so that socially beneficial outcomes can be identified.

Thirdly, we present technical results concerning the extent to which these properties can be satisfied. We show that although SW-JR is a natural extension of JR to the SCV setting, a committee satisfying SW-JR may not exist even under severe restrictions. Furthermore, checking whether there exists a committee satisfying SW-JR is NP-complete. The results always show that the more general setting SCV is considerably more challenging than approval-based committee voting. We then formalize a weakening of SW-JR called weak-SW-JR and present a polynomial-time algorithm that finds a committee that *simultaneously* satisfies weak-SW-JR and IW-JR. We also propose two natural generalizations of PAV (Proportional Approval Voting), a well-known rule for committee voting under approvals. However we show that neither of these two extensions satisfies both weak-SW-JR and IW-JR.

Sub-committee Voting

We propose a new setting called SCV that generalizes a number of voting models. The setting is a tuple $(N, C, \pi, q, \succsim)$

- $N = \{1, \ldots n\}$ is the set of voters/agents.

- $C = \{c_1, \ldots, c_m\}$ is the set of candidates.

- $\pi = \{C_1, \ldots, C_\ell\}$ is a partitioning of the candidates. Each C_j is referred to as a *candidate subset* from which a *sub-committee* is to be chosen.

- q is the quota function that specifies the number of candidates $q(C_j) = k_j$ to be selected from each subset C_j. We denote $\sum_{j=1}^{\ell} k_j$ by k.

- $\succsim = (\succsim_1, \ldots, \succsim_n)$ specifies for each agent i, her preferences/utilities over C. We allow the possibility that an agent does not compare candidates across candidate subsets. At a minimum it is required that each $\succsim_i$ is transitive and complete within each subset C_j, however additional restrictions can be introduced, as befitting the setting; they might even be replaced with cardinal utilities

An SCV outcome p specifies a real number $p(c)$ for each $c \in C$ with the following constraints:

$$0 \leq p(c) \leq 1 \quad \text{for all } c \in C$$
$$\sum_{c \in C} p(c) = k \quad \text{and} \quad \sum_{c \in C_j} p(c) = k_j \text{ for all } j \in \{1, \ldots, \ell\}.$$

In this paper we restrict our attention to discrete outcomes so that $p(c) \in \{0, 1\}$ but in general SCV can allow for probabilistic outcomes where $p(c)$ is the probability of selecting candidate c. For discrete outcomes, an outcome W will be a committee of size k that consists of ℓ sub-committees $W_1, \ldots, W_\ell$ where each $W_j \subseteq C_j$ and $|W_j| = k_j$.

If $\ell = 1$, $p(c) \in \{0, 1\}$ for all $c \in C$ and $k = 1$, we are in the *voting setting*. If $\ell = 1$, $p(c) \in \{0, 1\}$ for all $c \in C$, we are in the *committee/multi-winner voting setting* (Faliszewski et al. 2017). If $\ell = 1$ and $k = 1$, we are in the *probabilistic voting setting* (Gibbard 1977). If $p(c) \in \{0, 1\}$ for all $c \in C$ and $k_i = 1$ for all $i \in \{1, \ldots, \ell\}$, we are in the *public decision making setting* (Conitzer, Freeman, and Shah 2017). Note that public decision making setting is equivalent to the "voting on combinatorial domain" setting studied by Lang and Xia (2016). The latter setting allows for more complex preferences over the set of combinatorial outcomes but the expression of preferences is generally exponential in the number of candidates. If $p(c) \in \{0, 1\}$ for all $c \in C$ and $k_i = 1$ for all $i \in \{1, \ldots, \ell\}$ and $|C_j| = 2$ for all $j \in \{1, \ldots, \ell\}$, we are in the *multiple-referenda setting* (Brams, Kilgour, and Zwicker 1997; Lacy and Niou 2000; Çuhadaroğlu and Lainé 2012).
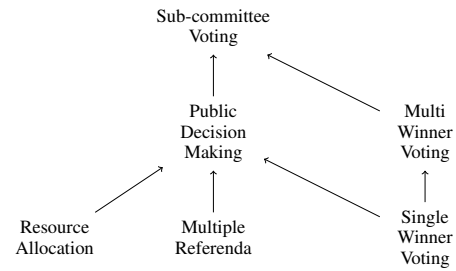

Figure 1: Relations between settings. An arrow from (A) to (B) denotes that setting (A) is a restriction of setting (B).

At a very abstract level, even a bicameral legislature can be viewed as an SCV setting in which $\ell = 2$. Even if there are no explicit multiple committees, there can be diversity constraints imposed on the committee that can be easily modelled as an SCV problem. For example, the problem of selecting five people with 3 women and 2 men can be viewed as an SCV problem with two candidate subsets.

In this paper, we will focus exclusively on approval-based voting in the SCV setting. In approval-based voting we replace each agent i's preference $\succsim_i$ with an approval ballot $A_i \subseteq C$ which represents the subset of candidates that she approves. The list $A = (A_1, \ldots, A_n)$ of approval ballots is referred to as the *ballot profile*. As per the general SCV setting introduced at the start of the section, the goal is to select a target k number of candidates from C which satisfy the quota function for each candidate subset.

Justified Representation in Approval-based Sub-committee Voting

We now focus on the SCV setting in which each agent approves a subset of the candidates. Based on the approvals, the goal is to identify a fair or representative outcome. Note that if $\ell = 1$, we are back in the committee voting setting. The approval-based SCV setting can be seen as capturing ℓ independent committee voting settings.

For the approval-based committee voting setting, justified representation (JR) is a desirable property.

Definition 1 (Justified representation (JR)). *Given a ballot profile $A = (A_1, \ldots, A_n)$ over a candidate set C and a target committee size k, we say that a set of candidates W of size $|W| = k$ satisfies justified representation for (A, k) if $\forall X \subseteq N$:*

$$|X| \geq \frac{n}{k} \text{ and } |\cap_{i \in X} A_i| \geq 1 \implies (|W \cap (\cup_{i \in X} A_i)| \geq 1)$$

One natural extension of JR to the case of SCV is to treat each candidate subset as an independent committee voting problem. Then an SCV outcome satisfies Intra-wise JR (IW-JR) if each sub-committee satisfies JR.

Definition 2 (Intra-wise JR (IW-JR)). *An SCV outcome W satisfies Intra-wise JR (IW-JR) if $\forall X \subseteq N$ and $\forall j \in \{1, \ldots, \ell\}$:*

$$|X| \geq \frac{n}{k_j} \text{ and } |(\cap_{i \in X} A_i) \cap C_j| \geq 1$$
$$\implies (|W \cap C_j \cap (\cup_{i \in X} A_i)| \geq 1)$$

We note that since a committee satisfying JR can always be attained by a polynomial-time algorithm (Aziz et al. 2017b), IW-JR is easy to achieve by treating each sub-committee voting as a separate committee voting problem.

The limitation of this approach is that it could be that each time the same voters are unrepresented in each sub-committee and they may ask for some representation in at least some sub-committee. Thus IW-JR can be considered as a 'local' JR axiom which ignores whether or not a given voter has already been represented in some other sub-committee.

In view of this limitation, another extension of JR to the case of SCV is to impose a JR-type condition across all sub-committees.[1] The definition of SW-JR is identical to the definition of JR for the committee voting setting.

Definition 3 (Span-wise JR (SW-JR)). *An SCV outcome W satisfies Span-wise JR (SW-JR) if $\forall X \subseteq N$:*

$$|X| \geq \frac{n}{k} \text{ and } |(\cap_{i \in X} A_i)| \geq 1 \implies (|W \cap (\cup_{i \in X} A_i)| \geq 1)$$

This approach to representation leads to a 'global' JR axiom which aims to represent large, cohesive groups of voters (i.e. $X \subseteq N : |X| \geq n/k$ and $\cap_{i \in X} A_i \neq \emptyset$) in some sub-committee, but not necessarily a sub-committee where they are cohesive.

The following example illustrates the IW-JR and SW-JR concepts.

Example 1. *Consider the SCV instance with $|N| = 6$,*

$$C = C_1 \cup C_2,$$

where $C_1 = \{a, b, c, d\}, C_2 = \{c_1, \ldots, c_5\}, k_1 = 2$ and $k_2 = 1$. Let the approval ballots of each voter be as follows

$$\begin{aligned}
A_i &= \{a, c_i\} && \text{if } i \in \{1, 2\} \\
A_i &= \{b, c_3\} && \text{if } i \in \{3, 4\} \\
A_5 &= \{b, c_4\} \\
A_6 &= \{c, d, c_5\}.
\end{aligned}$$

Note that $n/k_1 = 3, n/k_2 = 6$ and $n/k = 2$. An SCV outcome W satisfies IW-JR if and only if the group of voters $\{3, 4, 5\}$ are represented with at least one candidate they support from C_1 being included in W, i.e. $b \in W$.

Whilst, an SCV outcome W' satisfies SW-JR if and only if the group of voters $\{1, 2\}$ are represented with at least one candidate they support in either C_1 or C_2 being included in W', and the group of voters $\{3, 4\}$ also being represented with at least one candidate they support in either C_1 or C_2.

In particular, the committee $W = \{b, c, c_5\}$ satisfies IW-JR but not SW-JR, the committee $W' = \{a, c, c_3\}$ satisfies SW-JR but not IW-JR, and the committee $W'' = \{b, c, c_2\}$ satisfies both IW-JR and SW-JR.

Note that both SW-JR and IW-JR concern representation that are not at the level of single individual but at the level of large enough cohesive groups.[2]

Next we show that a SW-JR committee may not exist and is NP-hard to compute.

(Non)-existence and complexity of SW-JR committees

We show that a committee satisfying SW-JR may not exist under either of the two restriction (1) there are exactly two candidate subsets, and (2) $k_i = 1$ for all $i \in \{1, \ldots, \ell\}$.

Proposition 1. *A committee satisfying SW-JR may not exist even if there are exactly two candidate subsets and $k_i = 1$ for $i = 1, 2$.*

Proof. Consider an SCV instance where $|N| = n = 2, C = C_1 \cup C_2$ with $C_1 = \{a_1, a_2\}$ and $C_2 = \{b_1, b_2\}$, and $k_1 = k_2 = 1$. Note that $n/k = 1$.

If the approval ballots are $A_1 = \{a_1\}$ and $A_2 = \{a_2\}$, then there is no SCV outcome W (i.e. a committee) which satisfies SW-JR. This can be immediately observed since SW-JR requires both voters to be represented, however the quota $k_1 = 1$ prevents this from being possible. $\square$

The reader may note that above proof utilises an example where the voters have ballots which do not approve of any

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[1]Imposing representation requirements across all sub-committees implicitly assumes that the selections of all sub-committees are of comparable significance to the voters.

[2]In a related paper, Conitzer, Freeman, and Shah (2017) proposed fairness concepts for Public Decision Making that is equivalent to SCV in which $k_i = 1$ for each i. They considered different fairness notions that are based on proportional or envy-free allocations. The concepts involve viewing agents independently and are different from proportional representation concerns. When the number of sub-committees is less than the number of voters, the concepts they consider are trivially satisfied.

voter in some candidate subset (i.e. $A_i \cap C_2 = \emptyset$). This feature is not required to show the non-existence of an SW-JR committee however, it greatly simplifies the example.

Above we proved that a committee satisfying SW-JR may not exist. One could still aim to find such a committee whenever it exists. Next we prove that the problem of checking whether a SW-JR committee exists or not is NP-complete.

Proposition 2. *Checking whether an SW-JR committee exists or not is NP-complete.*

To show that checking whether a SW-JR committee exists is NP-complete we will reduce a given instance of the SET COVER problem, a known NP-complete problem (Garey and Johnson 1979), to an SCV instance - such that an SW-JR committee exists if and only if the SET COVER instance has a yes answer.

Below is a statement of the SET COVER problem.

SET COVER	
Input:	Ground set X of elements, a collection $\mathcal{L} = \{S_1, \ldots, S_t\}$ of subsets of X such that $X = \cup_{j=1}^{t} S_j$ and an integer k'.
Question:	Does there exist a $H \subset \mathcal{L}$ such that $|H| \leq k'$ and $X = \bigcup_{S_j \in H} S_j$

To build intuition for the formal proof, which follows, we provide an overview of the reduction.

The SET COVER problem involves answering whether or not there exists a collection of at most k' subsets $H \subseteq \mathcal{L}$ which cover another set X. This problem can be embedded into an SCV instance by letting the set X represent the set of voters and considering a candidate subset C_2 such that each element denotes an element of $\mathcal{L}$; that is,

$$C_2 = \{s_1, \ldots, s_t\}.$$

We then let each element of C_2, say s_j, be approved by voters $i \in N = X$ if and only if $i \in S_j$. By appropriately defining quota values and voter approval ballots on the remaining candidate subset C_1 it is shown that an SW-JR committee exists if and only if every voter is represented via a candidate in C_2 – this of course possible if and only if we have a yes-instance of the SET COVER problem

A formal proof is presented below.

Proof. We reduce a given SET COVER instance $(X, \mathcal{L}, k')$ to an SCV instance as follows: Let

$$N = X = \{1, \ldots, n\}$$
$$C = C_1 \cup C_2$$
$$= \{a_1, \ldots, a_n\} \cup \{s_1, \ldots, s_t\}$$

denote the set of voters, candidate set and partition into two candidate subsets. Let voter approval ballots be

$$A_i = \{s_j | i \in S_j\},$$

and $k_1 = n, k_2 = k'$. Without loss of generality we may assume that $t \geq k'$, also note that $k = n + k'$ and so $n/k < 1$.

Since every voter has a non-empty approval ballot (i.e. $N = X = \cup_{j=1}^{t} S_j$) and $n/k < 1$, a committee satisfies SW-JR if and only if every voter is represented.

If $(X, \mathcal{L}, k')$ is a yes-instance of the SET COVER problem then these exists a subset H with $|H| = k'$ such that $N = X = \bigcup_{S_j \in H} S_j$ – it follows that the committee

$$W = C_1 \cup \{s_i \,|\, S_i \in H\},$$

is a solution to the SCV problem and satisfies SW-JR. Conversely, if W is an SW-JR committee then the set

$$H = \{S_i \,|\, s_i \in W\},$$

provides a yes-instance to the SET COVER problem.

$\square$

Weak-SW-JR

In the previous section we showed that a committee satisfying SW-JR need not exist, and checking whether it does is NP-complete. Naturally and in the pursuit of a computationally tractable representation axiom we weaken the concept of SW-JR. In this section we present a weak version of SW-JR, appropriately referred to as weak-SW-JR. An SCV outcome satisfying weak-SW-JR is guaranteed to exist and is attainable via a polynomial-time algorithm.

Definition 4 (Weak-SW-JR). *An SCV outcome W satisfies* weak-Span-wise Justified Representation (weak-SW-JR) *if*

$$\forall X \subseteq N : |X| \geq \frac{n}{k} \text{ and } \left| \left(\bigcap_{i \in X} A_i \right) \cap C_j \right| \geq 1 \quad \text{for all } j$$
$$\implies \left| W \cap \left(\bigcup_{i \in X} A_i \right) \right| \geq 1$$

Informally speaking, the weak-SW-JR axiom captures the idea that if a "large", cohesive set of voters unanimously support at least one candidate in each candidate subset then they require representation in some sub-committee.

To illustrate the weak-SW-JR axiom we build on Example 1.

Example 2. *Recall the SCV instance presented in Example 1. An SCV outcome W satisfies weak-SW-JR if and only if the group of voters $\{3, 4\}$ are represented with at least one candidate they support in either C_1 or C_2, i.e. $W \cap \{b, c_3\} \neq \emptyset$. To contrast the previous concepts, the committee $W = \{c, d, c_3\}$ satisfies weak-SW-JR but neither SW-JR nor IW-JR.*

First observe that weak-SW-JR is indeed a (strict) weakening of the SW-JR concept.

Proposition 3. *SW-JR implies Weak-SW-JR. But weak-SW-JR does not imply SW-JR.*

The next proposition states that for a given SCV instance there may be three distinct committees W satisfying, respectively, weak-SW-JR but not IW-JR, IW-JR but not weak-SW-JR, and both weak-SW-JR and IW-JR simultaneously. That is, the weak-SW-JR and IW-JR representation axioms are distinct but are not mutually exclusive (the proof can be found in the supplement material).

Proposition 4. *Weak-SW-JR does not imply IW-JR and IW-JR does not imply weak-SW-JR. Also weak-SW-JR and IW-JR are not mutually exclusive concepts.*

Proof. We provide an example of an SCV instance which admits three distinct SCV outcomes which satisfy

1. weak-SW-JR but not IW-JR
2. IW-JR but not weak-SW-JR
3. both weak-SW-JR and IW-JR.

Consider the SCV instance with $|N| = 12$,

$$C = C_1 \cup C_2,$$

where $C_1 = \{c_1, \ldots, c_7\}$, $C_2 = \{a, b, c\}$, $k_1 = 1$ and $k_2 = 2$. Let the approval ballots of each voter be as follows

$$A_i = \begin{cases} \{c_i, a\} & \text{if } i \in \{1, 2, 3, 4, 5, 6\} \\ \{c_7, b\} & \text{if } i \in \{7, 8, 9, 10\} \\ \{c_8, c\} & \text{if } i \in \{11, 12\}. \end{cases}$$

First observe that for an SCV outcome W to satisfy weak-SW-JR the only set of voters which must be represented is $X = \{7, 8, 9, 10\}$ since they are a the only group of size greater or equal to $n/k = 4$ who unanimously support a voter in each of the candidate subsets i.e. $c_7 \in C_1$ and $b \in C_2$ are approved by every voter $i \in X$.

Whilst, for an SCV outcome to satisfy IW-JR it is required that the group $X' = \{1, 2, \ldots, 6\}$ are represented in C_2 by candidate $a \in C_2$. This is because X' is the only group of size greater or equal to $n/k_2 = 6$ who unanimously support a candidate in C_2.

Thus it follows immediately that there exists three distinct SCV outcomes which satisfy the three properties stated at the beginning of this proof, namely;

$$W^{\text{weak-SW-JR}} = \{c_1, b, c\}$$

satisfies weak-SW-JR but not IW-JR,

$$W^{\text{IW-JR}} = \{c_1, a, c\}$$

satisfies IW-JR but not weak-SW-JR and,

$$W^{\text{weak-SW-JR and IW-JR}} = \{c_7, a, c\}.$$

satisfies both IW-JR and weak-SW-JR. $\qquad\square$

An Algorithm for weak-SW-JR and IW-JR

The previous sections have introduced two appealing representation axioms, weak-SW-JR and IW-JR, which capture distinct notions of representation or fairness. The former axiom considers the structure of approvals across all candidate subsets, whilst the latter axiom considers each candidate subset as an independent event. This section will combine these two axioms and consider SCV outcomes W which satisfy both weak-SW-JR and IW-JR. We prove that a committee satisfying both weak-SW-JR and IW-JR is guaranteed to exists for every SCV setting and such a committee can be computed in polynomial-time.

We begin by presenting the following intermediate result before providing a constructive existence proof of a committee which satisfies weak-SW-JR and IW-JR.

Lemma 1. *Let $\{k_j\}_{j=1}^{\ell}$ be a sequence of positive numbers and $\{k_j'\}_{j=1}^{\ell}$ be a sequence of non-negative numbers. Then,*

$$\sum_{j=1}^{\ell} \frac{k_j'}{\sum_{i=1}^{\ell} k_i} \leq \max_{j \in [\ell]} \left\{ \frac{k_j'}{k_j} \right\}.$$

Proof. Let $M = \max_{j \in [\ell]} \left\{ \frac{k_j'}{k_j} \right\}$ and let $w_j = \frac{k_j}{\sum_i k_i}$, then

$$M = \sum_j w_j M \geq \sum_j w_j \frac{k_j'}{k_j} = \sum_j \frac{k_j'}{\sum_i k_i}, \quad \text{as required.}$$

$\square$

Input: SCV instance $(N, C, \pi, q, \succsim)$
Output: Committee W that satisfies both weak-SW-JR and IW-JR.

1. $W_j = \emptyset$ for $j = 1, \ldots, \ell$, $W = \bigcup_j W_j$
2. For each candidate subset $j = 1, \ldots, \ell$, allocate the candidates with support $\geq n/k_j$ from highest to lowest in support, removing the support of voters who are already represented in $W_j = W \cap C_j$.
3. From the remaining positions, consider unelected candidates who have support $\geq n/k$ among the unrepresented voters. Allocate these from highest to lowest in support removing the support of voters who are already represented in W.
4. If there are remaining positions allocate in any way.

Algorithm 1: Algorithm that returns a committee which satisfies both weak-SW-JR and IW-JR.

Proposition 5. *A committee which satisfies both weak-SW-JR and IW-JR always exists and can be attained via Algorithm 1.*

Proof. Consider Algorithm 1. We argue that the committee W returned by Algorithm 1 satisfies both weak-SW-JR and IW-JR.

First we show that IW-JR is satisfied. Suppose not, then during step 2 for some j we allocated k_j winning spots but failed to represent a group, say $X \subseteq N$, of size at least n/k_j who unanimously supports some candidate(s) in C_j. However, at each stage at least n/k_j additional voters are represented and so it must be the case that at least $k_j n/k_j = n$ voters were represented in W. That is, all voters have been represented which contradicts the existence of the set X. Thus IW-JR is always satisfied.

Now we show that weak-SW-JR is also satisfied. Suppose that after step 2, $|W_j| = k_j' < k_j$ for all j – if this were not the case then weak-SW-JR is trivially satisfied since all voters would then be represented in W. The proof now reduces to showing that there are enough 'places' left in W after allocating the $\sum_j k_j'$ places in Step 2.

In the 'worst case' every allocation for each j represents the same subset of voters – in this case W represents $\geq R :=$

$\max_j \{k'_j \frac{n}{k_j}\}$ voters with $\sum_j k'_j$ elected candidates. But then there are

$$\leq \frac{n-R}{n/k} = \frac{n - n\max_j\{\frac{k'_j}{k_j}\}}{n/k} = k\Big(1 - \max_j\big\{\frac{k'_j}{k_j}\big\}\Big)$$

possible mutually exclusive groups of size $\geq n/k$ which are unrepresented in W. In the worst case, each of these groups would unanimously support a different candidate in every candidate subset and so correspond to a problem set with respect to weak-SW-JR. Recall that we have $k - \sum_j k'_j$ winning spots left and so suffices to show that

$$k - \sum_j k'_j \geq k\Big(1 - \max_j \big\{\frac{k'_j}{k_j}\big\}\Big). \tag{1}$$

Note that we use the property that if problem set X exists they must unanimously support some candidate in every sub-committee and so we can ignore the quota issues. Finally, (1) follows immediately from dividing by k and applying Lemma 1. □

Testing Representation

Testing for SW-JR and IW-JR are easy given the polynomial-time testing of JR. Testing for weak-SW-JR is more involved, we conjecture the complexity is coNP-complete. Before we proceed, we outline the standard approval-based voting setting and an algorithm identified by Aziz et al. (2017b) to test JR.

Polynomial-time algorithm to verify JR

The standard setting of approval-based voting (AV) is a special case of SCV. In particular, an AV instance is a tuple (N, C, k, A) where N is a set of voters, C is the set of candidates, k is a positive integer and A is an approval ballot profile. An AV outcome (or committee) is a subset $W \subseteq C$ such that $|W| = k$. Note, that this is simply a special case of SCV when $\ell = 1$, $C_1 = C$ and $q(C_1) = k$.

The algorithm proposed by Aziz et al. (2017b) to test JR is as follows: given an Approval Voting instance (N, C, k, A) and outcome W, for each candidate $c \in C$ compute

$$s(c) = \big|\{i \in N \,:\, c \in A_i, A_i \cap W = \emptyset\}\big|.$$

The set W fails to provide JR for (A, k) if and only if there exists a candidate c with $s(c) \geq n/k$.

With minor modifications, the above algorithm provides a polynomial-time algorithm to test whether an SCV outcome satisfies SW-JR and IW-JR.

Proposition 6. *It can be checked in polynomial time whether a given committee satisfies SW-JR or not.*

Proposition 7. *It can be checked in polynomial time whether a given committee satisfies IW-JR or not.*

Generalizing PAV to SCV

In the setting of approval-based multi-winner voting, the Proportional Approval Voting (PAV) rule has been extensively studied and shown to satisfy many desirable representation properties. It has been shown in (Aziz et al. 2017b)

that PAV committees satisfy JR, though computing a PAV committee is NP-hard.[3]

Under PAV in the standard approval voting setting (AV), each voter who has j of their approved candidates in the committee W is assumed to derive utility of $r(j) := \sum_{p=1}^{j} 1/p$ if $j > 0$ and zero otherwise. The total utility of a committee W is then defined as $\text{PAV}(W) = \sum_{i \in N} r(|W \cap A_i|)$, this is known as the PAV-score. The PAV rule outputs the committee W of size k which maximizes the PAV-score among all committees of size k.

In this section we present two natural generalizations of the PAV rule to the SCV setting. This leads to two distinct PAV rules for the SCV setting. Neither proves useful is satisfying both of the key SCV representation axioms; weak-SW-JR and IW-JR. However, due to the success of the PAV rule in the standard committee voting settings is important discuss these variants.

Span-wise PAV (SW-PAV) is a generalization of the PAV rule which assumes voters gain utility solely from the number of their approved candidates in W. Thus, each voter i derives utility $r(|W \cap A_i|)$ and the SW-PAV score of a committee W is

$$\text{SW-PAV}(W) = \sum_{i \in N} r(|W \cap A_i|).$$

Intra-wise PAV (IW-PAV) is a generalization of the PAV rule which assumes voters gain utility from both the number of their approved candidates in W and also the diversity across sub-committees. Thus each voter i derives utility $\rho(|W \cap A_i|) := \sum_{j \in [\ell]} r(|W \cap A_i \cap C_j|)$, and the IW-PAV score of a committee W is

$$\text{IW-PAV}(W) = \sum_{i \in N} \rho(|W \cap A_i|) = \sum_{j \in [\ell]} \sum_{i \in N} r(|W \cap A_i \cap C_j|).$$

In both generalizations, a SW-PAV (IW-PAV) committee is defined to be a committee W satisfying the SCV quota conditions and maximizing the SW-PAV (IW-PAV) score.

To illustrate the distinction between SW-PAV and IW-PAV the following example is provided. Consider a voter i with approved and elected candidates $\{a, b, c\}$ such that $\{a, b\} \in C_1$ and $c \in C_2$. Then voter i's contribution to the SW-PAV score is $1\frac{5}{6}$ and her contribution to the IW-PAV score is $2\frac{1}{2}$.

Unfortunately, it can be shown that neither of these generalizations produces a committee which guarantees both the IW-JR and weak-SW-JR axioms. To be precise, IW-PAV satisfies IW-JR but need not satisfy weak-SW-JR and, in a similar fashion, SW-PAV satisfies weak-SW-JR but need not satisfy IW-JR. In the interest of space, however, we must omit these results.

Discussion

In this paper we formalized a general social choice model called sub-committee voting. We focussed on natural generalization of JR from the approval-based committee voting setting to the approval-based SCV setting. Some of the

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[3]In fact PAV is viewed as one of the most compelling rules for approval-based committee voting because it satisfies EJR a property stronger than JR (Aziz et al. 2017b).

results are summarized in Table 1. It will be interesting to consider generalizations of stronger versions of justified representation such as PJR and EJR. For example, IW-JR can straightforwardly be strengthened to IW-PJR or IW-EJR.

It will be interesting to consider more general preferences that need not be approval-based. Several research questions that have been intensely studied in subdomains of SCV apply as well to SCV. For example, it will be interesting to extend axioms and rules for single-winner or multi-winner voting to that of SCV.

	Representative committee exists	Complexity of computing
SW-JR	No	NP-c
IW-JR	Yes	in P
weak-SW-JR	Yes	in P
IW-JR & weak-SW-JR	Yes	in P

Table 1: Properties of justified representation concepts for sub-committee voting.

In any combinatorial setting, one can view the voting process as either simultaneous voting or sequential voting (Barrot and Lang 2016; Lang and Xia 2016; Freeman, Zahedi, and Conitzer 2017b; 2017a). We formalized SCV as a static model in which ℓ sub-committees are to be selected simultaneously. The representation notions that we formalized can also be considered if voting over each sub-committee is conducted sequentially over time. The axioms that we consider such as SW-JR apply as well to understand the quality of an outcome in these online or sequential settings.

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